Constructing strictly plurisubharmonic functions on the complexification of certain non-compact manifolds.

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Abstract

The purpose of this paper is to extend a strictly convex function f on M to a strictly plurisubharmonic function $\hat{f}$ on its complexification $\mathfrak{D}_M$.

If there existed a π-invariant strictly plurisubharmonic function in $\mathfrak{D}_M$, its restriction to M must be strictly convex.

When M is a symmetric space of non-compact type, we show that the π-invariant lifting of any strictly convex function in M is strictly plurisubharmonic in $\mathfrak{D}_M$. As a byproduct, strictly plurisubharmonic exhaustions of the disk bundle T^rM can be constructed in an explicit way.

1 Introduction

The purpose of this paper is to extend a strictly convex function f on M to a strictly plurisubharmonic function $\hat{f}$ on its complexification $\mathfrak{D}_M$.

Let (M, g) be a complete real-analytic Riemannian manifold and let $\mathfrak{D}_M \subset TM$ be the maximal domain so that the adapted complex structure, characterized by turning Riemann foliations $\gamma^{\mathbb{C}}$ into holomorphic curves, is defined. This complex structure has brought out a naturally defined strictly plurisubharmonic function, namely, the potential function $\rho(x, v) := |v|^2$ which has controlled the vertical growth of $\mathfrak{D}_M$.

When the base manifold M is compact, the function $-\log(r^2 - \rho)$ can be used as a strictly plurisubharmonic exhaustion of the disk bundle T^rM. This function is no longer an exhaustion when M is non-compact; the growth along the zero section M is out of control.

Is it possible to construct a strictly plurisubharmonic function to get hold the growth along M? The first observation is, Lemma 3.1, that if there existed a π-invariant strictly plurisubharmonic function in $\mathfrak{D}_M$, its restriction to M must be strictly convex. A natural question arisen then is: Suppose there existed a strictly convex function in M, is it possible to extend it to a strictly plurisubharmonic function

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in $\mathfrak{D}_M$? One difficulty is there is in general no explicit way to write down the adapted complex structure away from M; it is not clear how to associate the Levi-Civita connection $\tilde{\nabla}$ in $\mathfrak{D}_M$ with the Levi-Civita connection ∇ in M.

However, when M is a symmetric space of non-compact type, the complex structure along Riemann foliations can be explicitly written down in terms of Jacobi fields and root decomposition. We have obtained the main result, Theorem 5.1, of this article saying that the π-invariant lifting $\hat{f}$ of a strictly convex function f in M is strictly plurisubharmonic in $\mathfrak{D}_M$ when M is a symmetric space of non-compact type.

Performing linear combinations of $\hat{f}$ with the potential fucniton ρ, both horizontal and vertical growth can be controlled and strictly plurisubharmonic functions will be constructed to exhaust Grauert tubes T^rM. As a byproduct, we show T^rM is a Stein manifold.

The organization of this article is as follows. In §2 we review some properties, needed for later development, of the adapted complex structure. We also recall the definition of convex functions in Riemannian manifolds. §3 is devoted to characterizing the strictly plurisubharmonicity of π-invariant functions in some local situations. In §4 the computation has been reduced to holomorphic vector fields along Riemann foliations. Root decompositions and Jacobi fields in symmetric spaces of non-compact type are discussed in §5. By the help of complexified Jacobi fields, the existence theorem: the π-lifting of a strictly convex function in M is a strictly plurisubharmonic function in $\mathfrak{D}_M$ is proved in this section. Finally, in §6, concrete strictly plurisubharmonic exhaustion functions for T^rM are constructed. As a byproduct, the Steinness of T^rM for any $r \leq r_{max}(M)$ is shown for non-compact symmetric M.

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2 A complex structure on TM and convex functions in M

2.1 Properties of the adapted complex structure

Let $\gamma : \mathbb{R} \to M$ be a geodesic in a complete real-analytic Riemannian manifold (M, g). The adapted complex structure is a unique complex structure in a maximal domain $\mathfrak{D}_M \subset TM$ such that the immersion $\gamma^{\mathbb{C}} : \mathbb{C} \cap \gamma^{\mathbb{C}^{-1}}(\mathfrak{D}_M) \to \mathfrak{D}_M$ defined by $\gamma^{\mathbb{C}}(t + is) = (\gamma(t), s\dot{\gamma}(t)) \in \mathfrak{D}_M$ is a holomorphic mapping.

The adapted complex structure can be described in terms of Jacobi fields, cf. [Le-Sz], as follows. Given $p \in M$ and an orthnormal basis $\{v_1, \cdots, v_n\}$ of T_pM, Jacobi fields $\{\xi_j, \eta_j\}_{j=1}^n$ along the geodesic $\gamma(t) := \exp_p tv_1$ are uniquely determined by the following initial conditions:

$$\begin{aligned} \xi_j(0) = v_j, \ \ \xi_j'(0) = 0; \\ \eta_j(0) = 0, \ \ \eta_j'(0) = v_j. \end{aligned} \tag{2.1}$$

Fixing $j \in \{1, \cdots, n\}$, there exist a 1-parameter family of geodesics $\{\gamma_\sigma^j : -\epsilon < \sigma < \epsilon\}$ with $\gamma_0^j = \gamma$ such for any $t \in \mathbb{R}$,

$$\xi_j(t) = \frac{\partial}{\partial \sigma} \gamma_\sigma^j(t)|_{\sigma=0}. \tag{2.2}$$

In other words, $\{\gamma_\sigma^j : -\epsilon < \sigma < \epsilon\}$ is a variational family of geodesics deciding the Jacobi field $\xi_j(t)$ along the geodesic $\gamma(t)$.

Let $N_s : TM \to TM$ be the multiplication by s on fibers. The notation $\gamma_\sigma^j(t+is)$ will be used to denote the point

$$\gamma_\sigma^j(t+is) := N_s \dot{\gamma}_\sigma^j(t) = (\gamma_\sigma^j(t), s\dot{\gamma}_\sigma^j(t)) \in TM. \tag{2.3}$$

Jacobi fields $\xi_j(t)$ in M along $\gamma(t)$ can be extended to vector fields $\xi_j(t+is)$ in $\mathfrak{D}_M$ along the Riemann foliation $\gamma^{\mathbb{C}}(t+is)$ by setting

$$\xi_j(t+is) = \frac{\partial}{\partial \sigma} \gamma_\sigma^j(t+is)|_{\sigma=0}. \tag{2.4}$$

A family of vector fields $\{\eta_j(t+is) : j = 1, \cdots, n\}$ along $\gamma^{\mathbb{C}}(t+is)$ can be constructed in a similar way. It was proved in [Le-Sz, Prop. 6.4] that away from M, $\{\xi_j^{(1,0)} := \xi_j - iJ\xi_j\}_{j=1}^n$ are $\mathbb{C}$-independent holomorphic vector fields in $\mathfrak{D}_M$ along $\gamma^{\mathbb{C}}$.

The Riemannian metric g defines the Levi-Civita connection on TM and splits the tangent space $T_z(TM)$ into the direct sum of the vertical subspace $T_z^v(TM)$ and the horizontal subspace $T_z^h(TM)$. In [Le-Sz], the canonical Kähler metric on $\mathfrak{D}_M$ is assigned by, for any $X, Y \in T_z(TM)$,

$$h(X,Y) = \frac{1}{2}\{\langle (JX)_h, Y_v \rangle - \langle (JX)_v, Y_h \rangle\} \tag{2.5}$$

where Y_v, resp. Y_h, denotes the vertical, resp. horizontal, component of Y and $\langle \cdot, \cdot \rangle$ is the Riemannian metric on $T_{\pi(z)}M$. For later application, we summarize the following two facts as a lemma and give a brief explanation to it.

Lemma 2.1. *(1). JX contains a non-zero horizontal, resp. vertical, component if $X \neq 0$ is vertical, resp. horizontal.*

(2). $\eta_j(is)$ is a vertical vector in $T_{\gamma^{\mathbb{C}}(is)}(TM)$.

Proof. Suppose both X and JX are horizontal, then $h(X,X) = 0$, a contradiction to the fact that h is a metric on $\mathfrak{D}_M$.

Let $\{\tilde{\gamma}_\sigma^j : -\epsilon < \sigma < \epsilon\}$ be a variational family of geodesics deciding the Jacobi field $\eta_j(t)$ along the geodesic $\gamma(t)$. Then

$$0 = \eta_j(0) = \frac{\partial}{\partial \sigma} \tilde{\gamma}_\sigma^j(0)|_{\sigma=0}. \tag{2.6}$$

By (2.4),

$$\eta_j(is) = \frac{\partial}{\partial \sigma} \tilde{\gamma}_\sigma^j(is)|_{\sigma=0} = \frac{\partial}{\partial \sigma} (\tilde{\gamma}_\sigma^j(0), s\dot{\tilde{\gamma}}_\sigma^j(0))|_{\sigma=0} \tag{2.7}$$

has contained no horizontal component. $\square$

2.2 Convex functions and strictly plurisubharmonic functions

The Hessian for a C^2 function f in a Riemannian manifold (M, g) is defined as, for any C^∞ vector fields X, Y in M and any point $p \in M$,

$$\begin{aligned} D^2 f(X,Y)(p) &= X_p(Yf) - (\nabla_{X_p} Y) f \\ &= X_p(Yf) - \langle \nabla_{X_p} Y, grad\ f(p) \rangle \end{aligned} \tag{2.8}$$

where ∇ is the covariant differentiation associated to the Riemannian metric. A function f is said to be (strictly) *convex* at p if its Hessian at p is (positive) semi-positive definite.

Clearly $D^2 f(X,Y)(p)$ has depended only on the vectors X_p and Y_p. Hence $D^2 f$ can be considered as a symmetric bilinear form on the tangent space $T_p M$. This local property has actually implied that a C^2 function f in M is convex if and only if $(f \circ \gamma)'' \geq 0$ for any geodesic γ.

The existence of convex functions has long associated with non-positively curved manifolds. It is well known that in a non-positively curved manifold, the distance square function $d^2(\cdot, p)$ from any fixed point p is strictly convex (*c.f.* [Bi-O'N]). In [Ak], the author has established an 1-1 correspondence between convex functions on a symmetric space M and convex functions on a Cartan subalgebra.

A C^2 function f in a complex manifold Ω is *strictly plurisubharmonic* at the point p if the Levi form $Lf(W, W)(p)$ is strictly positive for any holomorphic vector field W in Ω.

Since a Kähler manifold is a Riemannian manifold equipped with a complex structure compatible with the Riemannian metric, Greene-Wu, [Gr-Wu], p.14, have furnished an interpretation between these two convexities. They showed that a C^2 function f in a Kähler manifold Ω is (strictly) plurisubharmonic if and only if

$$D^2 f(X,X) + D^2 f(JX, JX) \tag{2.9}$$

is (strictly) positive for any vector field X in Ω.

3 π-invariant lifting from M to $\mathfrak{D}_M$

Notice that the projection $\pi : TM \to M$ is a smooth mapping and the Kähler structure of $\mathfrak{D}_M$ is determined by the Riemannian structure of (M, g).

Throughout this paper, we will consider functions in $\mathfrak{D}_M$ invariant under π, *i.e.,* $f(z) = f(w)$ provided $\pi(z) = \pi(w)$. We will check the regularity first. The smoothness of π has implied that if a π-invariant function f is C^∞ in $\mathfrak{D}_M$ then its restriction $f|_M$ is C^∞ in M. On the other hand, a C^∞ function h in M may be lifted to a C^∞ function $\hat{h}$ in $\mathfrak{D}_M$ by setting

$$\hat{h}(z) = h(\pi(z)),\ z \in \mathfrak{D}_M. \tag{3.1}$$

The function $\hat{h}$ in $\mathfrak{D}_M$ is called the *π-invariant lifting* of h in M. We are hoping that the π-invariant lifting of strictly convex functions in M might produce strictly plurisubharmonic functions in $\mathfrak{D}_M$.

First of all, there is some constrain on the existence of π-invariant strictly plurisubharmonic functions.

Lemma 3.1. *Let f be a π-invariant (strictly) plurisubharmonic function in $\mathfrak{D}_M$. Then the restriction function $f|_M$ is (strictly) convex on M.*

Proof. Let γ be a geodesic in M then $\gamma^{\mathbb{C}} \cap \mathfrak{D}_M$ is a flat Riemann surface in $\mathfrak{D}_M$. The π-invariance have implied that $f|_{\gamma^{\mathbb{C}}} = f|_{\gamma^{\mathbb{C}}}(t+is) = f|_{\gamma^{\mathbb{C}}}(t)$ where t is the arc length of the geodesic γ from some fixed point.

Being a (strictly) plurisubharmonic function in $\mathfrak{D}_M$, its restriction to the Riemann surface $\gamma^{\mathbb{C}}$ is (strictly) subharmonic. That is,

$$0 \leq (\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}) f|_{\gamma^{\mathbb{C}}} = \frac{\partial^2 f|_{\gamma}}{\partial t^2} = (f|_M \cdot \gamma)''. \tag{3.2}$$

The convexity of $f|_M$ is concluded.

□

Bishop-O'Neil, [Bi-O'N], have classified some obstructions to the existence of a convex function in a complete Riemannian manifold M: (1) non-constant convex functions only exist on manifolds of infinite volume; (2) if M admits a convex function f, then f is constant on each closed geodesic. Following the characterization of Bishop-O'Neil, we have

Corollary 3.2. *There exists no non-constant π-invariant plurisubharmonic function in $\mathfrak{D}_M$ if M contained a closed geodesic or M had finite volume.*

From now on, we only consider Riemannian manifolds M admitting strictly convex functions. The following local property will be shown first.

Lemma 3.3. *Let h be a strictly convex function in M. Then its π-invariant lifting $\hat{h}$ is strictly plurisubharmonic in a neighborhood $\mathfrak{M}$ of M in $\mathfrak{D}_M$.*

Proof. Given $q \in M$, pick a geodesic normal coordinate $\{x_1, \cdots, x_n\}$ in a neighborhood U_q of q in M. Then the adapted complexification in a small neighborhood $\mathfrak{M}_q$ of U_q in $\mathfrak{D}_M$ has complex coordinates $\{z_1 = x_1 + \sqrt{-1}y_1, \cdots, z_n = x_n + \sqrt{-1}y_n\}$. By the π-invariance, $\hat{h}(z) = \hat{h}(x + \sqrt{-1}y) = h(x)$ in $\mathfrak{M}_q$ and

$$\frac{\partial^2 \hat{h}}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 h}{\partial x_i \partial x_j}, \ \forall i, j. \tag{3.3}$$

Since h is strictly convex and $\{x_1, \cdots, x_n\}$ is a geodesic normal coordinate centered at q, the Hessian $\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right) = \left(\frac{\partial^2 \hat{h}}{\partial z_i \partial \bar{z}_j}\right)$ is positive definite at q. By the smoothness of $\hat{h}$, it is clear that $\hat{h}$ is strictly plurisubharmonic in some neighborhood of q in $\mathfrak{D}_M$. Shrinking $\mathfrak{M}_q$ if necessary, we may assume $\hat{h}$ is strictly plurisubharmonic in $\mathfrak{M}_q$. The set $\mathfrak{M} := \cup_{q \in M} \mathfrak{M}_q$ has settled the proof. □

It is known that the minimum set of a strictly convex function is a single point and this point is the unique critical point.

Let p be the critical point of the strictly convex function $f : M \to \mathbb{R}$. We first consider the behavior of the lifting over a small neighborhood U of p in M. The restriction of the Kähler structure of $\mathfrak{D}_M$ and the restriction of the Levi-Civita connection $\tilde{\nabla}$ of $\mathfrak{D}_M$ to $TU \cap \mathfrak{D}_M$ have produced a Kähler structure and a Levi-Civita connection $\tilde{\nabla}$ on $TU \cap \mathfrak{D}_M$.

Lemma 3.4. *Let f be a real-analytic strictly convex function on M with the critical point p, and $\hat{f}$ be its π-invariant lifting to $\mathfrak{D}_M$. There exists a neighborhood U of p in M so that $\hat{f}$ is a smooth strictly plurisubharmonic function in $TU \cap \mathfrak{D}_M$.*

Proof. Taking a geodesic normal coordinate $\{x_1, \cdots, x_n\}$ centered at p, in a neighborhood U of p in M. Since f is real-analytic, strictly convex and has p as the critical point it has the following convergent power series expansion. For $x \in U$,

$$f(x) = f(p) + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \text{ higher order terms}, \tag{3.4}$$

where (c_{ij}) is a positive-definite $n \times n$ matrix.

Let $z_0 = (p, v) \in TU \cap \mathfrak{D}_M$ and let $W = U \times \mathcal{O}$ be a neighborhood of z_0 in $TU \cap \mathfrak{D}_M$ where $\mathcal{O}$ is an open set in $\mathbb{R}^n$ with a coordinate system $\{y_1, \cdots, y_n\}$ centered at $v \in T_p U$.

The π-invariance shows that in terms of the coordinates $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$, the function $\hat{f}$ has the following local expression in $TU \cap \mathfrak{D}_M$,

$$\hat{f}(x, y) = f(p) + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \text{ higher order terms in } x. \tag{3.5}$$

Since $\hat{f}$ has no first order terms and $p = \{x_j = 0 : j = 1, \cdots, n\}$, the vector field

$$grad\ \hat{f}(0, y) = \sum_{j=1}^{n} \frac{\partial \hat{f}(0, y)}{\partial x_j} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial \hat{f}(0, y)}{\partial y_j} \frac{\partial}{\partial y_j} = 0,\ \forall y \in \mathcal{O}. \tag{3.6}$$

Let X be a vector field in $TU \cap \mathfrak{D}_M$. Then for $z_0 = (p, v)$,

$$\begin{aligned} & D^2 \hat{f}(X, X)(z_0) + D^2 \hat{f}(JX, JX)(z_0) \\ &= X_{z_0}(X\hat{f}) - \langle \tilde{\nabla}_{X_{z_0}} X, grad\ \hat{f}(z_0) \rangle + JX_{z_0}(JX\hat{f}) - \langle \tilde{\nabla}_{JX_{z_0}} JX, grad\ \hat{f}(z_0) \rangle \\ &= X_{z_0}(X\hat{f}) + JX_{z_0}(JX\hat{f}). \end{aligned} \tag{3.7}$$

For $z = (x, y) \in TU \cap \mathfrak{D}_M$, we may assume

$$X_z = \sum_{j=1}^{n} \alpha_j(x, y) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \beta_j(x, y) \frac{\partial}{\partial y_j}. \tag{3.8}$$

As $z_0 = (0, 0)$ in the coordinate system $\{x, y\}$,

$$X_{z_0}(X\hat{f}) = \sum_{j,k=1}^{n} c_{jk} \alpha_j(0, 0) \alpha_k(0, 0). \tag{3.9}$$

Since the matrix (c_{ij}) is positive-definite, (3.9) is strictly positive unless $\alpha_j^2(0,0) = 0$, $\forall j$. Similarly, $JX_{z_0}(JX\hat{f})$ is strictly positive unless JX_{z_0} has contained no component from $\frac{\partial}{\partial x_j}, j = 1, \cdots, n$.

Suppose $\alpha_j^2(0,0) = 0$ for all j, then $X_{z_0} = \sum_{j=1}^n \beta_j(0,0)\frac{\partial}{\partial y_j}$ is a vertical vector in $T_{z_0}(TM)$. By Lemma 2.1, JX_{z_0} must contain some horizontal component, *i.e.,* JX_{z_0} must contain some vector $\frac{\partial}{\partial x_j}$. Then either $X_{z_0}(X\hat{f})$ or $JX_{z_0}(JX\hat{f})$ has to be strictly positive.

By (3.7), we conclude that $D^2\hat{f}(X,X)(z_0) + D^2\hat{f}(JX,JX)(z_0) > 0$. By continuity, $D^2\hat{f}(X,X) + D^2\hat{f}(JX,JX) > 0$ in some neighborhood of z_0 in $\mathfrak{D}_M$. Shrinking U if necessary, we conclude that $\hat{f}$ is strictly plurisubharmonicin $TU \cap \mathfrak{D}_M$. $\square$

Fixing a point $p \in M$, it is clear that the distance square $d^2(x,p)$ is a continuous function in M. Generically, this function fails to be differentiable. For instance, in a compact Riemannian manifold the function $d^2(x,p)$ is not differentiable at the cut locus $C(p)$. Nevertheless, it behaves very well near p since a Riemannian manifold is locally asymptotic to the Enclidean space. In a small neighborhood of p, the function $f(x) := d^2(x,p)$ is smooth and strictly convex. It is real-analytic provided (M,g) is.

We have the following corollary on the behavior of Riemann surfaces in $\mathfrak{D}_M$.

Corollary 3.5.

1. *Let $f(x) := d^2(x,p)$ be the distance function on the real-analytic (M,g), and $\hat{f}$ be its π-invariant lifting to $\mathfrak{D}_M$. There exists a neighborhood U of p in M so that $\hat{f}$ is a smooth strictly plurisubharmonic function in $TU \cap \mathfrak{D}_M$.*

2. *A Riemann surface $\mathcal{S} \subset \mathfrak{D}_M$ has intersected the fibers T_qM, for any $(q,v) \in \mathcal{S}$, transversally. That is, for any $(q,v) \in \mathcal{S}$, $T_{(q,v)}\mathcal{S} \not\subseteq T_qM$.*

Proof. The first part is clear from Lemma 3.4 and the fact that f is real-analytic and strictly convex in U.

Since $\mathcal{S}$ is an one-dimensional complex submanifold of $\mathfrak{D}_M$ there exists a complex coordinate $\{z_1, \cdots, z_n\}$ around $(q,v) \in \mathcal{S}$ so that $\mathcal{S}$ is locally generated by z_1 and $\bar{z}_1$. Let $h_q(x) := d^2(x,q)$ and let $\hat{h}_q$ be its π-invariant lifting to $\mathfrak{D}_M$. By *1.*, $\hat{h}_q$ is strictly plurisubharmonicin in $TU \cap \mathfrak{D}_M$ for some neighborhood U. Restricted the function $\hat{h}_q$ to $\mathcal{S}$, then $\frac{\partial^2 \hat{h}_q}{\partial z_1 \partial z_{\bar{1}}} > 0$ near (q,v).

If $T_{(q,v)}\mathcal{S} \subset T_qM$, there exists a neighborhood U' of (q,v) in $\mathcal{S}$ so that $U' \subset T_qM$. Then $\frac{\partial^2 \hat{h}_q}{\partial z_1 \partial z_{\bar{1}}}(q,v) = 0$, a contradiction. $\square$

Lemma 3.4 has settled the strictly plurisubharmonic lifting near the critical point. Away from the critical point, the non-vanishing of some first order terms has made the situation much complicated.

Let f be a real-analytic strictly convex function on (M,g) with the critical point p. Given $q \in M$, $q \neq p$, we take a geodesic normal coordinate $\{x_1, \cdots, x_n\}$ centered

at q in a neighborhood U of q in M. By the real-analyticity, f has the following convergent power series expansion, for $x \in U$

$$f(x) = f(q) + \sum_{j=1}^{n} a_j x_j + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \text{ higher order terms} \tag{3.10}$$

where (c_{ij}) is a positive-definite $n \times n$ matrix. Since f has no other critical point than p, some $a_j \neq 0$.

Instead of the global consideration, we now restrict f to U and consider functions h and Q defined in this coordinate chart as follows. Let

$$h(x) = \sum_{j=1}^{n} a_j x_j \tag{3.11}$$

and let

$$Q(x) = f(x) - h(x) = f(q) + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \text{ higher order terms} \tag{3.12}$$

be real-analytic functions defined in U.

Let $W = U \times \mathcal{O}$ be a product neighborhood of (q, v) in $TU \cap \mathfrak{D}_M$ where $\mathcal{O}$ is the largest open set in $\mathbb{R}^n$ such that $U \times \mathcal{O} \subset TU \cap \mathfrak{D}_M$. In $\mathcal{O}$, pick a coordinate system $\{y_1, \cdots, y_n\}$ centered at $v \in T_q U$. It is clear that $\mathfrak{D}_U = TU \cap \mathfrak{D}_M$ is an open domain in $\mathfrak{D}_M$ and the π-invariant liftings are

$$\hat{f} = \hat{h} + \hat{Q}. \tag{3.13}$$

For any vector field X in $\mathfrak{D}_U$,

$$\begin{aligned} D^2\hat{f}(X,X) + D^2\hat{f}(JX,JX) =& D^2\hat{h}(X,X) + D^2\hat{h}(JX,JX) \\ &+ D^2\hat{Q}(X,X) + D^2\hat{Q}(JX,JX). \end{aligned} \tag{3.14}$$

Since Q has contained no first order terms, its π-invariant lifting $\hat{Q}$ on $TU \cap \mathfrak{D}_M$ behaves quite well.

Lemma 3.6. *$\hat{Q}$ is strictly plurisubharmonic along the fiber $T_q M \cap \mathfrak{D}_M$.*

Proof. $Q(x)$ is written in geodesic normal coordinates centered at q with a positive definite (c_{ij}), it is a real-analytic strictly convex function in U with the critical point q. Lemma 3.4 can be applied to this situation and the lemma is proved. The argument only relies on the point q, it works for the whole fiber $T_q M \cap \mathfrak{D}_M$. $\square$

It remains to show the $\hat{h}$ part. The Hessian to the vector field X is,

$$D^2\hat{h}(X,X)(z) = X_z X \hat{h} - \langle \tilde{\nabla}_{X_z} X, grad\ \hat{h}(z) \rangle. \tag{3.15}$$

Unlike all the above discussed cases, the vanishing of the gradient term is no longer guaranteed since $\hat{h}(z)$ has contained some first order terms. This non-vanishing

gradient has made it difficult to compute (3.15). The question to be answered is the following:

Question: Let h be a linear function in the normal geodesic coordinate chart $U \subset M$. Is its π-invariant lifting $\hat{h}$ plurisubharmonic on $TU \cap \mathfrak{D}_M$? (3.16)

Without further structure imposed in M, it is hard to tell how much the question (3.16) can be true. With the help of the root description in the coming section, an affirmative answer to it will be provided for symmetric spaces of non-compact type.

4 Reducing to holomorphic vector fields along $\gamma^{\mathbb{C}}$

Showing $\hat{h}$ is plurisubharmonic at the point $(p, v) \in TU \cap \mathfrak{D}_M$ is equivalent to finding holomorphic vector fields $\{Z_j : j = 1, \cdots, n\}$, $\mathbb{C}$-independent at (p, v), in $TU \cap \mathfrak{D}_M$ so that the matrix $(Z_k \bar{Z}_j \hat{h}(q, v))$ is semi-positive definite.

In fact, it is sufficient to consider holomorphic vector fields along the Riemannian surface $\gamma^{\mathbb{C}}$ containing the point (q, v). Let X be a holomorphic vector field along a Riemann surface $\gamma^{\mathbb{C}} \ni (q, v)$. Locally, we may extend X to a holomorphic vector field $\tilde{X}$ in a neighborhood of $\gamma^{\mathbb{C}}$ in $\mathfrak{D}_M$ as following. Since $\gamma^{\mathbb{C}}$ is an 1-dimensional complex submanifold of $\mathfrak{D}_M$ there exists, in a neighborhood $\mathfrak{N}$ of (q, v) in $\mathfrak{D}_M$, holomorphic coordinates $\{z_1, \cdots, z_n\}$ so that $\gamma^{\mathbb{C}} \cap \mathfrak{N}$ is represented by the z_1-axis. In terms of these coordinates, the holomorphic vector field X along $\gamma^{\mathbb{C}} \cap \mathfrak{N}$ is expressed as

$$X(z_1) = \sum_{k=1}^{n} f_k(z_1) \frac{\partial}{\partial z_k} \tag{4.1}$$

for some holomorphic functions f_k in $\gamma^{\mathbb{C}} \cap \mathfrak{N}$. Let $\tilde{f}_k(z) = \tilde{f}_k(z_1, \cdots, z_n) := f_k(z_1)$ be the extension of f_k from $\gamma^{\mathbb{C}} \cap \mathfrak{N}$ to $\mathfrak{N}$. Then

$$\tilde{X}(z) := \sum_{k=1}^{n} \tilde{f}_k(z) \frac{\partial}{\partial z_k} \tag{4.2}$$

is a holomorphic vector field in $\mathfrak{N}$ and $\tilde{X}|_{\gamma^{\mathbb{C}}} = X$. For any function φ in $\mathfrak{N}$ and $(q, v) \in \gamma^{\mathbb{C}} \cap \mathfrak{N}$, the following calculation shows it is sufficient to take holomorphic vector fields along $\gamma^{\mathbb{C}}$ into consideration.

Lemma 4.1. *Let $X(z_1) = \sum_{k=1}^{n} f_k(z_1) \frac{\partial}{\partial z_k}$ and $Y(z_1) = \sum_{k=1}^{n} g_k(z_1) \frac{\partial}{\partial z_k}$ be two holomorphic vector fields along $\gamma^{\mathbb{C}} \cap \mathfrak{N}$. Let $\tilde{X}(z)$ and $\tilde{Y}(z)$ be their extensions to $\mathfrak{N}$. Then*

$$\tilde{X}\bar{\tilde{Y}}\varphi(q, v) = X\bar{Y}\varphi(q, v).$$

Proof. A direct calculation shows:

$$\begin{aligned}
\tilde{X}\bar{\tilde{Y}}\varphi(q,v) &= \sum_{k,l=1}^{n} \tilde{f}_k(q,v)\overline{\tilde{g}_l(q,v)}\frac{\partial^2\varphi}{\partial z_k \partial \bar{z}_l}(q,v) \\
&= \sum_{k,l=1}^{n} f_k(q,v)\overline{g_l(q,v)}\frac{\partial^2\varphi}{\partial z_k \partial \bar{z}_l}(q,v) \\
&= X\bar{Y}\varphi(q,v).
\end{aligned} \tag{4.3}$$

□

We summarize the above discussions as a lemma.

Lemma 4.2. *A C^2 real-valued function f is plurisubharmonic at the point (q,v) if the $n \times n$ matrix*

$$[(\xi_k - iJ\xi_k)(\xi_j + iJ\xi_j)f(q,v)] \geq 0 \tag{4.4}$$

where $\{\xi_j - iJ\xi_j\}_{j=1}^n$ are holomorphic vector fields, $\mathbb{C}$-independent at (q,v), along the Riemann foliation $\gamma^{\mathbb{C}}$ from the geodesic $\gamma(t) = \exp_q t\frac{v}{|v|}$.

5 Plurisubharmonicity of the π-lifting over symmetric spaces of non-compact type

In this section, we will give an affirmative answer to the question (3.16) for certain kind of Riemannian manifolds. Jacobi fields on Riemannian symmetric spaces have already been used in [Sz] and [Da-Sz]. Following the Akhiezer-Gindikin's approach, *c.f.* [Ak-Gi], of using the root system to describe the canonical complexification, Halverscheid, [Ha] and [B-H-H], has been able to write down these Jacobi fields along $\gamma^{\mathbb{C}}$ more precisely in terms of the root system.

Let M be a Riemannian symmetric space of non-compact type. Let G be the identity component of the isometry group of M and $K \subset G$ be the identity component of the isotropy group at $q \in M$. Then M can be written as $M = G/K$. Let $\mathfrak{g}$ be the Lie algebra of G; $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the corresponding Cartan decomposition. Then the tangent space T_qM can be identified with $\mathfrak{m}$. A positive definite scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ may be assigned as following: $\langle X, W\rangle = -B(X,W)$ for $X, W \in \mathfrak{k}$; $\langle X, W\rangle = 0$ for $X \in \mathfrak{k}, W \in \mathfrak{m}$; $\langle X, W\rangle = (X,W)_g$ for $X, W \in \mathfrak{m}$, where $(\ ,\ )_g$ is the Riemannian metric of the symmetric space M and B is the killing form on $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. For any $H \in \mathfrak{a}$, the adjoint operator $adH : \mathfrak{g} \to \mathfrak{g}$ is symmetric linear and adH has commuted with adK for any $K \in \mathfrak{a}$. Since they commute, all the operators $adH, H \in \mathfrak{a}$ have shared the same eigenvectors. A decomposition according to common eigenvectors is thus obtained:

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}^{\alpha} \tag{5.1}$$

where $Z(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ and $\{\alpha : \mathfrak{a} \to \mathbb{R} | \alpha \in \Lambda\}$ are called the non-zero *roots* of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ with

$$\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g} : [A, X] = \alpha(A)X, \forall A \in \mathfrak{a}\}. \tag{5.2}$$

It is understood that a root can appear repeatedly and $\dim \mathfrak{g}^{\alpha} = 1$ for any $\alpha \in \Lambda$. In [Ha] and [B-H-H], the symmetric operator $-(adH)^2 : \mathfrak{m} \to \mathfrak{m}$ is considered to achieve the following root decomposition of $\mathfrak{m}$ w.r.t. $\mathfrak{a}$:

$$\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda} \{w - \theta(w) | w \in \mathfrak{g}^{\alpha}\} \tag{5.3}$$

where θ is an involution of $\mathfrak{g}$ so that $\mathfrak{k}$ is the 1-eigenspace and $\mathfrak{m}$ is the (-1)-eigenspace. The eigenvalue of $-(adH)^2$ on $w - \theta(w), w \in \mathfrak{g}^{\alpha}$, is $-(\alpha(H))^2 \leq 0$. If $\dim \mathfrak{m} = n, \dim \mathfrak{a} = l$, we may arrange the roots $\{\alpha_1, \cdots, \alpha_n\}$ so that $\alpha_1 = \cdots = \alpha_l = 0$.

Fixing an $H \in \mathfrak{a}$, let $\gamma(t) = exp_q\ tH$ be the geodesic on M determined by H. A common orthonormal eigenbasis $\{v_1, \cdots, v_n\}$ of $\{-(adK)^2 : K \in \mathfrak{a}\}$ in $\mathfrak{m}$ has determined $2n$ Jacobi fields along the geodesic $\gamma(t)$:

$$\begin{aligned} \xi_j(t) &= cosh(\alpha_j(H)t)v_j(t); \\ \eta_j(t) &= \frac{1}{\alpha_j(H)} sinh(\alpha_j(H)t)v_j(t) \\ &= \frac{1}{\alpha_j(H)} tanh(\alpha_j(H)t)\xi_j(t) \end{aligned} \tag{5.4}$$

where $v_j(t)$ is the parallel transportation of v_j along $\gamma(t)$. When $\alpha_j(H) = 0, \xi_j(t) = v_j(t)$ and $\eta_j(t) = tv_j(t)$. They can be extended to $\mathbb{C}$-independent holomorphic vector fields $\{\xi_j^{(1,0)}(t+is)\}_{j=1}^n$ and $\mathbb{C}$-independent holomorphic vector fields $\{\eta_j^{(1,0)}(t+is)\}_{j=1}^n$ along $\gamma^{\mathbb{C}}$.

The relation $\eta_j^{(1,0)}(t) = \frac{1}{\alpha_j(H)} tanh(\alpha_j(H)(t))\xi_j^{(1,0)}(t)$ and the uniqueness of the complexification along $\gamma^{\mathbb{C}}$, *c.f.* [Ha], yields

$$\eta_j^{(1,0)}(t+is) = \frac{1}{\alpha_j(H)} tanh(\alpha_j(H)(t+is))\xi_j^{(1,0)}(t+is). \tag{5.5}$$

Separating the real and the imaginary parts with

$$\begin{aligned} f_j(t+is) &= \frac{1}{\alpha_j(H)} tanh((t+is)\alpha_j(H)), \quad \text{if } \alpha_j(H) \neq 0; \\ f_j(t+is) &= t+is, \quad \text{if } \alpha_j(H) = 0. \end{aligned} \tag{5.6}$$

Then, along $\gamma^{\mathbb{C}}$

$$J\xi_j(t+is) = \frac{1}{\Im f_j(t+is)} [\eta_j(t+is) - \Re f_j(t+is)\xi_j(t+is)]. \tag{5.7}$$

Notice that the adapted complex structure is defined up to those $\gamma^{\mathbb{C}}(t+is)$ so that $\frac{1}{\Im f_j(t+is)}$ is well-defined.

Since a Riemannian symmetric space of non-compat type has non-positive sectional curvature, it is known that the distance square from a fixed point is strictly convex. Notice that this distance function is not $Isom(M)$-invariant. Other K_p, the isotropy group at $p \in M$, invariant convex functions on M are associated with convex functions in the Cartan subalgebra, *cf.* [Ak].

Theorem 5.1. *Let M be a Riemannian symmetric space of non-compat type. Let f be a strict convex function on M, then its π-invariant lifting $\hat{f}$ is a strictly plurisubharmonic function on $\mathfrak{D}_M$.*

Proof. Since the (strict) plurisubharmonicity, respectively the (strict) convexity, is a local property, it is sufficient to consider it at a point $(q, v) \in T_qM \cap \mathfrak{D}_M$. We may identify the tangent space T_qM with $\mathfrak{m}$ from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Once this (q, v) is chosen, a standard theorem, *c.f.* [He, theorem 6.2], has guaranteed that there exists a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{m}$ so that $\frac{v}{|v|} := v_1 \in \mathfrak{a}$. We will adapt the notation $v = \mu v_1$. An orthonormal common eigenbasis $\{v_1, \cdots, v_n\}$ of $\{-(adK)^2 : K \in \mathfrak{a}\}$ in $\mathfrak{m} \cong T_qM$ can be chosen.

Let U be a neighborhood of q in M equipped with the geodesic normal coordinates $\{x_1, \cdots, x_n\}$ obtained from the orthonormal basis $\{v_1, \cdots, v_n\}$ at T_qM. Restrict the function f to U and consider its power series expansion $f(x) = h(x) + Q(x)$ as done in (3.10)–(3.12), where $h(x) = \sum_{j=1}^n c_j x_j$ is consisted of all the first order terms and the π-invariant lifting of $\hat{f}$ to $TU \cap \mathfrak{D}_M$ is $\hat{f} = \hat{h} + \hat{Q}$.

Since f is strictly convex, it is shown in Lemma 3.6 that $\hat{Q}$ is strictly plurisubharmonic at (q, v). It is then left to show $\hat{h} = \sum_{j=1}^n c_j x_j$ is plurisubharmonic at $(q, v) \in T_qM$.

Let $\{\xi_j(t), \eta_j(t)\}_{j=1}^n$ be the $2n$ Jacobi fields along the geodesic $\gamma(t) = exp_q tv_1$ decided by the initial conditions

$$\begin{aligned} \xi_j(0) = v_j,\ \dot{\xi}_j(0) = 0; \\ \eta_j(0) = 0,\ \dot{\eta}_j(0) = v_j. \end{aligned} \tag{5.8}$$

The explicit solutions of such Jacobi fields along the geodesic $\gamma(t)$ are given at (5.4) and the corresponding holomorphic vector fields $\xi_j^{(1,0)}(t + is), \eta_j^{(1,0)}(t + is)$ along the Riemann foliation $\gamma^{\mathbb{C}}(t + is) \cap \mathfrak{D}_M$ are described at (5.5)–(5.7). Notice that the point $(q, v) := \gamma^{\mathbb{C}}(i\mu)$.

To complete the proof, the following computations are needed.

Lemma 5.2. $\eta_j(t + is)\hat{h} = \frac{1}{\alpha_j(v_1)} sinh(\alpha_j(v_1)t)c_j$.

Proof. Let $\{\tilde{\gamma}_\sigma^j : \sigma \in (-\epsilon, \epsilon)\}$ be a variation of geodesics in M with $\tilde{\gamma}_0^j = \gamma$ and the Jacobi field η_j along γ is determined by $\eta_j(t) = \frac{d}{d\sigma}\tilde{\gamma}_\sigma^j(t)|_{\sigma=0}$. By definition,

$$\eta_j(t + is) = \frac{d}{d\sigma}(\tilde{\gamma}_\sigma^j(t), s\dot{\tilde{\gamma}}_\sigma^j(t))|_{\sigma=0}. \tag{5.9}$$

Then,

$$\begin{aligned}
\eta_j(t+is)\hat{h} &= \lim_{\sigma\to 0}\frac{\hat{h}(\tilde{\gamma}^j_\sigma(t), s\dot{\tilde{\gamma}}^j_\sigma(t)) - \hat{h}(\gamma^j_0(t), s\dot{\gamma}^j_0(t))}{\sigma} \\
&= \lim_{\sigma\to 0}\frac{\hat{h}(\tilde{\gamma}^j_\sigma(t), 0) - \hat{h}(\tilde{\gamma}^j_0(t), 0)}{\sigma} \\
&= \eta_j(t)h \\
by (5.4) \quad &= \frac{1}{\alpha_j(v_1)} sinh(\alpha_j(v_1)t)c_j.
\end{aligned} \tag{5.10}$$

□

Lemma 5.3. $\xi_j(t+is)\hat{h} = cosh(\alpha_j(v_1)t)c_j$.

Proof. Let $\{\gamma^j_\sigma : \sigma \in (-\epsilon, \epsilon)\}$ be a variation of geodesics in M with $\gamma^j_0 = \gamma$ and the Jacobi field ξ_j along γ is determined by $\xi_j(t) = \frac{d}{d\sigma}\gamma^j_\sigma(t)|_{\sigma=0}$. By definition,

$$\xi_j(t+is) = \frac{d}{d\sigma}(\gamma^j_\sigma(t), s\dot{\gamma}^j_\sigma(t))|_{\sigma=0}. \tag{5.11}$$

Then,

$$\begin{aligned}
\xi_j(t+is)\hat{h} &= \lim_{\sigma\to 0}\frac{\hat{h}(\gamma^j_\sigma(t), s\dot{\gamma}^j_\sigma(t)) - \hat{h}(\gamma^j_0(t), s\dot{\gamma}^j_0(t))}{\sigma} \\
&= \lim_{\sigma\to 0}\frac{\hat{h}(\gamma^j_\sigma(t), 0) - \hat{h}(\gamma^j_0(t), 0)}{\sigma} \\
&= \xi_j(t)h \\
by (5.4) \quad &= cosh(\alpha_j(v_1)t)c_j.
\end{aligned} \tag{5.12}$$

□

Lemma 5.4. *For any j and k,*

$$\xi_k\xi_j\hat{h}(q,v) = 0; \quad J\xi_k\xi_j\hat{h}(q,v) = 0.$$

Proof.

$$\begin{aligned}
\xi_k\xi_j\hat{h}(q,v) &= \xi_k(i\mu)\xi_j(t+is)\hat{h} \\
&= \xi_k(i\mu)cosh(\alpha_j(v_1)t)c_j \\
&= \xi_k(i)N^*_\mu cosh(\alpha_j(v_1)t)c_j \\
&= \xi_k(i)cosh(\alpha_j(v_1)t)c_j.
\end{aligned} \tag{5.13}$$

By the choice of $\xi_j(0) = v_j$, the definition (5.11) has implied that $\xi_j(i)$, for $j = 2, \cdots, n$, is not tangent to $\gamma^{\mathbb{C}}$. Since $cosh(\alpha_j(v_1)t)$ is a function along $\gamma(t)$, it is clear that $\xi_k(i)cosh(\alpha_j(v_1)t) = 0$ for $k \neq 1$. On the other hand,

$$\xi_1(i)cosh(\alpha_j(v_1)t) = \frac{d}{dt}cosh(\alpha_j(v_1)t)|_{t=0} = 0. \tag{5.14}$$

It is concluded that $\xi_k\xi_j\hat{h}(q,v) = 0$ for any j and k.

By (5.7) and the fact that $\Re f_j(is) = 0$ for all $s \in \mathbb{R}$, $J\xi_j(i\mu) = \frac{1}{\Im f_j(i\mu)}\eta_j(i\mu)$.

$$\begin{aligned} J\xi_k\xi_j\hat{h}(q,v) &= J\xi_k(i\mu)cosh(\alpha_j(v_1)t)c_j \\ &= \frac{1}{\Im f_j(i\mu)}\eta_j(i)N_\mu^* cosh(\alpha_j(v_1)t)c_j \\ &= 0 \end{aligned} \tag{5.15}$$

since $\eta_j(i)$, by Lemma 2.1, only effects fiber directions. $\square$

Lemma 5.5. *For any j and k, $J\xi_k J\xi_j\hat{h}(q,v) = 0$.*

Proof. The computation is separated into 3 parts.

$$J\xi_k J\xi_j\hat{h}(q,v) = J\xi_k(i\mu)\cdot\frac{1}{\Im f_j(t+is)}[\eta_j(t+is) - \Re f_j(t+is)\xi_j(t+is)]\hat{h}. \tag{5.16}$$

$$\begin{aligned} -J\xi_k(i\mu)\cdot\frac{\Re f_j(t+is)\xi_j(t+is)}{\Im f_j(t+is)}\hat{h} &= \frac{-cosh(\alpha_j(v_1)t)c_j}{\Im f_j(i\mu)}|_{t=0}\ J\xi_k(i\mu)\cdot\Re f_j(t+is)|_{t=0} \\ &= \frac{-cosh(\alpha_j(v_1)t)c_j}{(\Im f_j(i\mu))^2}|_{t=0}\ \eta_k(i\mu)\cdot\Re f_j(t+is)|_{t=0} \\ &= \frac{-cosh(\alpha_j(v_1)t)c_j}{(\Im f_j(i\mu))^2}|_{t=0}\ \eta_k(i)N_\mu^*\Re f_j(t+is)|_{t=0} \\ &= 0, \end{aligned} \tag{5.17}$$

since $\eta_k(i)$ only works on fiber directions and $\Re f_j(is) \equiv 0$.

$$\begin{aligned} & J\xi_k(i\mu)\cdot\frac{1}{\Im f_j(t+is)}\eta_j(t+is)\hat{h}|_{t=0} \\ &= \frac{\eta_j(i\mu)}{\Im f_j(i\mu)}\ \frac{sinh(\alpha_j(v_1)t)c_j}{\alpha_j(v_1)\Im f_j(t+is)}|_{t=0} \\ &= 0, \end{aligned} \tag{5.18}$$

since both $sinh(0) = 0$ and $\eta_j(i\mu)sinh(\alpha_j(v_1)t) = 0$ hold. $\square$

Let's denote $Z_k = \xi_k - iJ\xi_k$. Since $\hat{h}$ is real-valued, it is clear that the matrix $[Z_k\bar{Z}_j\hat{h}(q,v)]$ is Hermitian symmetry where

$$Z_k\bar{Z}_j = \xi_k\xi_j + J\xi_k J\xi_j - iJ\xi_k\xi_j + i\xi_k J\xi_j. \tag{5.19}$$

Lemma 5.4 along with Lemma 5.5 has implied that $[Z_k\bar{Z}_j\hat{h}(q,v)] = [i\xi_k J\xi_j\hat{h}(q,v)]$. By the Hermitian symmetry,

$$-\xi_j J\xi_k\hat{h}(q,v) = \overline{\xi_k J\xi_j\hat{h}(q,v)}. \tag{5.20}$$

Lemma 5.6. $Z_k\bar{Z}_j\hat{h}(q,v) = 0$ *for any j and k.*

Proof. It is equivalent to showing $\xi_k J\xi_j \hat{h}(q,v)=0$ for any j and k.

$$\begin{aligned}\xi_k J\xi_j \hat{h}(q,v) &= \xi_k(i\mu)\cdot\frac{1}{\Im f_j(t+is)}[\eta_j(t+is)-\Re f_j(t+is)\xi_j(t+is)]\hat{h}\\ &= \xi_k(i\mu)\frac{\eta_j\hat{h}}{\Im f_j}-\xi_k(i\mu)\frac{\Re f_j}{\Im f_j}\xi_j\hat{h}\\ &= \frac{c_j}{\alpha_j(v_1)\Im f_j(i\mu)}\xi_k(i)N_\mu^* sinh(\alpha_j(v_1)t)|_{t=0}\\ &\quad -\frac{cosh(\alpha_j(v_1)t)c_j}{\Im f_j(i\mu)}|_{t=0}\ \xi_k(i)N_\mu^*\Re f_j|_{t=0}.\end{aligned}\tag{5.21}$$

It is clear that except for $\{\xi_1 J\xi_j\hat{h}(q,v):j=1,\cdots,n\}$ all the other $\xi_k J\xi_j\hat{h}(q,v)$ have vanished since $\xi_k(i)$ is not tangent to $\gamma^{\mathbb{C}}$ when $k\neq 1$. By (5.20), $\xi_1 J\xi_j\hat{h}(q,v)=-\overline{\xi_j J\xi_1\hat{h}(q,v)}=0$ when $j\neq 1$.

The only possibly non-vanishing term is $\xi_1 J\xi_1\hat{h}(q,v)$. Since $v_1\in\mathfrak{a}$, $\alpha_1(v_1)=0$. We have $f_1(t+is)=t+is$ and

$$\begin{aligned}&\frac{1}{\alpha_j(v_1)}\xi_1(i)N_\mu^* sinh(\alpha_j(v_1)t)|_{t=0}=1;\\ &cosh(\alpha_j(v_1)t)\xi_1(i)N_\mu^*\Re f_j|_{t=0}=1.\end{aligned}\tag{5.22}$$

That is, $\xi_1 J\xi_1\hat{h}(q,v)=0$ as well. We conclude that all $\xi_k J\xi_j\hat{h}(q,v)=0$. □

The decomposition $\hat{f}=\hat{h}+\hat{Q}$ along with Lemmas 3.6 and 5.6 has reached the conclusion that $\hat{f}$ is strictly plurisubharmonic at the point (q,v). Since $\mathfrak{D}_M$ is foliated by Riemann surfaces of the form $\gamma^{\mathbb{C}}$, the above argument can be applied to any point in $\mathfrak{D}_M$. Thus, the strictly plurisubharmonicity of the function $\hat{f}$ on $\mathfrak{D}_M$ is concluded. □

6 Strictly plurisubharmonic exhaustions of T^rM

An r-disk bundle equipped with the adapted complex structure is called a Grauert tube T^rM of radius r over M,

$$T^rM:=\{(x,v)\in T_xM:|v|=r\}\subset\mathfrak{D}_M.\tag{6.1}$$

For each real-analytic Riemannian manifold, there associated to M a non-negative real number $r_{max}(M)$ indicating the maximal possible radius for T^rM to exist, *i.e.*, $T^sM\not\subseteq\mathfrak{D}_M$ for any $s>r_{max}(M)$. The potential function $\rho(x,v):=|v|^2$ is strictly plurisubharmonic in $\mathfrak{D}_M$ which has controlled the vertical growth of the domain.

Since a symmetric space is co-compact, the Steinness of $T^rM, 0<r\leq r_{max}$, can be concluded from the fact that the universal covering of a Stein manifold is Stein, [Ka].

In this section, we will construct a precise and concrete family of exhaustions to show T^rM is Stein when M is a non-compact symmetric space.

Let $f(x) := dist^2(x,p)$ be the distance square to a fixed point $p \in M$ and let $\hat{f}$ be its π-invariant lifting to $\mathfrak{D}_M$.

It is clear that f is strictly convex when M is of non-compact symmetry. Following Theorem 5.1 the funciton $\hat{f}$ is strictly plurisubharmonic in $\mathfrak{D}_M$, which has controlled the growth along the zero section M. In fact, there is a two-parameters' family of strictly plurisubharmonic functions in $\mathfrak{D}_M$ taking care of both the horizontal and the vertical growth: for any $s, t > 0$, the functions $s\,\rho(z) + t\,\hat{f}(z)$ are strictly plurisubharmonic in $\mathfrak{D}_M$. We are looking for suitable s and t to achieve an exhaustion.

Since M has non-positive sectional curvature, $r_{max}(M) < \infty$. Fixed a $0 < r \leq r_{max}(M) < \infty$, for any $s > \max\{1, \frac{1}{r^2}\}$ the function

$$\eta_s(z) := \rho(z) + \frac{\hat{f}(z)}{s} \tag{6.2}$$

is strictly plurisubharmonic in $\mathfrak{D}_M$ and the domain

$$\begin{aligned} \Omega_s^r :&= \{z \in \mathfrak{D}_M : \eta_s(z) < r^2 - \frac{1}{s}\} \\ &= \{(x,v) \in \mathfrak{D}_M : |v|^2 + \frac{f(x)}{s} < r^2 - \frac{1}{s}\} \end{aligned} \tag{6.3}$$

is clearly a strictly pseudoconvex bounded domain in T^rM. Further properties of domains Ω_s^r are listed in the following.

Lemma 6.1. *For any $t, s > \max\{1, \frac{1}{r^2}\}, 0 < r \leq r_{max}(M)$, the domain Ω_s^r have the following properties.*

1. $\Omega_s^r \Subset T^rM$;
2. Ω_s^r *is a Stein manifold;*
3. $\Omega_s^r \subset \Omega_t^r$ *whenever $s < t$;*
4. $\lim_{s\to\infty} \Omega_s^r = T^rM$.

Proof. 1. is clear from the boundedness of Ω_s^r : $(x,v) \in \Omega_s^r \Rightarrow f(x) < sr^2$ and $|v| < r^2$.

2. Having a strictly plurisubharmonic exhaustion function $-\log(r^2 - \frac{1}{s} - \eta_s)$, the domain Ω_s^r is a Stein manifold.

3. For $z = (x,v) \in \Omega_s^r$,

$$|v|^2 + \frac{f(x)}{s} < r^2 - \frac{1}{s}. \tag{6.4}$$

Thus, for any $t > s$,

$$|v|^2 + \frac{f(x)}{t} < r^2 - \frac{1}{s} < r^2 - \frac{1}{t}. \tag{6.5}$$

Therefore $z \in \Omega_t^r$.

4. Since every Ω_s^r sits in T^rM, $\lim_{s\to\infty}\Omega_s^r \subset T^rM$. We now show $T^rM \subset \lim_{s\to\infty}\Omega_s^r$. Given $(x,v) \in T^rM$ with $|v| = \mu < r$ and $f(x) = l^2$. We would like to find a $\beta > 0$ such that $(x,v) \in \Omega_\beta^r$. It amounts to finding β such that

$$\mu^2 + \frac{l^2}{\beta} < r^2 - \frac{1}{\beta}, \tag{6.6}$$

which is equivalent to $\beta > \frac{l^2+1}{r^2-\mu^2}$. Thus, $T^rM \subset \lim_{s\to\infty}\Omega_s^r$ and 4. is concluded. $\square$

In other words, T^rM is exhausted by an increasing family of relatively compact Stein manifolds. The next question is whether the increasing limit of a family of Stein manifolds is a Stein? A counterexample has been provided by Fornaess [Fo] where he has constructed an increasing family of 3-dimensional Stein manifolds each biholomorphic to the ball such that the limit is not Stein. Nevertheless, when the index set is dense, Docquier and Grauert, *c.f.* [D-G] , have proved the limit is Stein as well.

Thus, T^rM is Stein if M is a symmetric space of non-compact type.

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